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# Quantum Deformations of Generalized Kac–Moody Algebras and Their Modules

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We construct quantum groups $U_q(\mathfrak{g})$ associated with generalized Kac–Moody algebras $\mathfrak{g}$ with admissible Borcherds–Cartan matrices. We also construct quantum deformations of highest weight modules over $U(\mathfrak{g})$ with integral highest weights. We show that, for generic $q$, Verma modules over $U(\mathfrak{g})$ with integral highest weights and irreducible highest weight modules over $U(\mathfrak{g})$ with dominant integral highest weights can be deformed to those over $U_q(\mathfrak{g})$ in such a way that the dimensions of weight spaces are invariant under the deformation. In particular, for generic $q$, the characters of irreducible highest weight modules over $U_q(\mathfrak{g})$ with dominant integral highest weights are given by the Weyl–Kac–Borcherds formula.


## INTRODUCTION

The *quantum groups* were introduced independently by Drinfel'd [Dr] and Jimbo [J] in their study of the quantum Yang–Baxter equation and two-dimensional solvable lattice models. The quantum group $U_q(\mathfrak{g}_0)$ associated with a symmetrizable Kac–Moody algebra $\mathfrak{g}_0$ is a certain deformation of the universal enveloping algebra $U(\mathfrak{g}_0)$ of $\mathfrak{g}_0$. That is, it is a family of Hopf algebras whose structure tends to that of $U(\mathfrak{g}_0)$ as $q \to 1$. It turns out that the quantum groups provide the fundamental algebraic structure behind many branches of mathematics and mathematical physics such as the representation theory of Kac–Moody algebras, solvable lattice models in statistical mechanics, quantum field theory, and invariant theory of links and 3-manifolds.

In [Lu1], for generic $q$, it was shown that integrable highest weight modules over a symmetrizable Kac–Moody algebra $\mathfrak{g}_0$ can be deformed to

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those over the corresponding quantum group $U_q(\mathfrak{g}_0)$ and that the characters of integrable highest weight modules over $U_q(\mathfrak{g}_0)$ are invariants for generic $q$. Therefore, if an integrable highest weight module over $U_q(\mathfrak{g}_0)$ has a simple structure for some special value of $q$, we can calculate the character for that value of $q$. In the language of solvable lattice models, the quantum parameter $q$ corresponds to the absolute temperature, and, in particular, $q = 0$ corresponds to the absolute temperature zero. So we can expect that the quantum groups $U_q(\mathfrak{g}_0)$ and their modules have rather simple structure when $q = 0$. This speculation led Kashiwara to develop the *crystal base theory* for the quantum groups $U_q(\mathfrak{g}_0)$ associated with symmetrizable Kac–Moody algebras $\mathfrak{g}_0$ [Kas1–Kas6]. He constructed certain natural bases, called the *global crystal bases*, for integrable highest weight modules over $U_q(\mathfrak{g}_0)$ and for the subalgebra $U_q(\mathfrak{g}_0^-)$, which have a surprisingly simple structure at $q = 0$. On the other hand, using a geometric approach, Lusztig independently constructed the *canonical bases* with very nice properties [Lu2–Lu4]. In [GL], it was shown that the canonical bases coincide with the global crystal bases.

*Crystal bases* can be thought of as the global crystal bases (or the canonical bases) at $q = 0$, and they have an extremely nice behavior with respect to the tensor product. They provide us with a powerful combinatorial tool to investigate the structure of integrable highest weight modules over $U_q(\mathfrak{g}_0)$, and they have a lot of beautiful applications. For instance, in [KMN1, KMN2], we developed the theory of *affine crystals* and, using the *path realization* of crystal bases for integrable highest weight modules over quantum affine Lie algebras $U_q(\mathfrak{g}_0)$, we obtained the closed expression of the *one-point function* in solvable lattice models in terms of string functions which arise from the characters of integrable highest weight modules over $U_q(\mathfrak{g}_0)$.

On the other hand, in his study of vertex algebras [B1, FLM] and Monstrous moonshine [CN], Borcherds introduced a new class of infinite dimensional Lie algebras called *generalized Kac–Moody algebras* [B2]. The structure and the representation theories of generalized Kac–Moody algebras are very similar to those of Kac–Moody algebras, and many basic facts about Kac–Moody algebras can be generalized to generalized Kac–Moody algebras [B2–B5, HMY, Ju1, Ju2, N1–N5]. But there are some striking differences, too. For example, generalized Kac–Moody algebras may have *imaginary simple roots* with norms $\leq 0$ whose multiplicity can be $> 1$, which is not the case for Kac–Moody algebras. Also, we still do not have the complete reducibility theorem for the integrable modules in the category $\mathcal{O}$.

For unitarizable irreducible highest weight modules over symmetrizable generalized Kac–Moody algebras, Borcherds proved a character formula [B2], called the *Weyl–Kac–Borcherds formula*, which generalizes the

Weyl–Kac formula for integrable highest weight modules over symmetrizable Kac–Moody algebras [K]. As in the case of Kac–Moody algebras, the Weyl–Kac–Borcherds formula, when applied to the one-dimensional trivial representation, yields the *denominator identity*. In [Ka1], using the Weyl–Kac–Borcherds formula and the denominator identity, we obtained a closed-form root multiplicity formula for all symmetrizable generalized Kac–Moody algebras, which enables us to study the structure of a symmetrizable generalized Kac–Moody algebra $\mathfrak{g}$ as a representation of a Kac–Moody algebra $\mathfrak{g}_0$ contained in $\mathfrak{g}$. One of the most interesting examples of generalized Kac–Moody algebras is the *monster Lie algebra*, which was constructed by Borcherds in his proof of the Moonshine conjecture [B5]. In [Ka], applying our root multiplicity formula to the monster Lie algebra, we obtained some interesting relations for the coefficients $c(n)$ of the elliptic modular function

$$j(q) - 744 = \sum_{n \geq -1} c(n) q^n = q^{-1} + 196884q + 21493760q^2 + \cdots .$$

In this paper, we construct quantum groups $U_q(\mathfrak{g})$ associated with generalized Kac–Moody algebras $\mathfrak{g}$ with admissible Borcherds–Cartan matrices. We also construct quantum deformations of highest weight modules over $U(\mathfrak{g})$ with integral highest weights. We show that, for generic $q$, Verma modules over $U(\mathfrak{g})$ with integral highest weights and irreducible highest weight modules over $U(\mathfrak{g})$ with dominant integral highest weights can be deformed to those over $U_q(\mathfrak{g})$ in such a way that the dimensions of weight spaces are invariant under the deformation. In particular, for generic $q$, the characters of irreducible highest weight modules over $U_q(\mathfrak{g})$ with dominant integral highest weights are given by the Weyl–Kac–Borcherds formula.

This work was greatly inspired by [Lu1], and most of our exposition follows the framework of [Lu1]. But on several critical occasions, the arguments in [Lu1] cannot be generalized to our setting in a straightforward way, mainly due to the following reasons: (i) the size of Borcherds–Cartan matrices may be infinite; (ii) the diagonal entries of Borcherds–Cartan matrices are not necessarily 2; (iii) the complete reducibility theorem for the integrable modules in the category $\mathcal{O}$ is not available in our setting. On these occasions, either new arguments for the proofs are introduced, or the arguments in [Lu1] are modified. As the results of [Lu1] initiated a lot of important advances in the representation theory of quantum groups $U_q(\mathfrak{g}_0)$, we expect that our work will be followed by a series of interesting developments in the representation theory of quantum groups $U_q(\mathfrak{g})$ associated with generalized Kac–Moody algebras $\mathfrak{g}$. For instance, we expect to see the developments of crystal base theory, canonical basis theory, and generalizations of Lakshmibai–Seshadri bases ([La1, La2, Li1, Li2]).

## 1. GENERALIZED KAC–MOODY ALGEBRAS

Let $\mathbf{F}$ be a field of characteristic 0, and let $\mathbf{K}$ be a subfield of $\mathbf{F}$ which is isomorphic to a subfield of $\mathbf{R}$, the field of real numbers. Let $I$ be a countable (possibly infinite) index set. A matrix $A = (a_{ij})_{i,j \in I}$ with $a_{ij} \in \mathbf{K}$ is called a *Borcherds–Cartan matrix* if it satisfies: (i) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$; (ii) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbf{Z}$ if $a_{ii} = 2$; (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. Let $I^{\mathrm{re}} = \{i \in I \mid a_{ii} = 2\}$, $I^{\mathrm{im}} = \{i \in I \mid a_{ii} \leq 0\}$, and let $\underline{m} = (m_i \mid i \in I)$ be a collection of positive integers such that $m_i = 1$ for all $i \in I^{\mathrm{re}}$. We call $\underline{m}$ the *charge* of the matrix $A$. A Borcherds–Cartan matrix $A$ is said to be *symmetrizable* if there is a diagonal matrix $D = \operatorname{diag}(s_i \mid i \in I)$ with $s_i > 0$ $(i \in I)$ such that $DA$ is symmetric. In this paper, we assume that $A$ is symmetrizable.

Definition 1.1. The *generalized Kac–Moody algebra* $\mathfrak{g} = \mathfrak{g}(A, \underline{m})$ with a symmetrizable Borcherds–Cartan matrix $A$ of charge $\underline{m} = (m_i \mid i \in I)$ is the Lie algebra over $\mathbf{F}$ generated by the elements $h_i$, $d_i$ $(i \in I)$, $e_{ik}$, $f_{ik}$ $(i \in I,\ k = 1, \ldots, m_i)$ with the defining relations:

$$\begin{aligned}
&[h_i, h_j] = [h_i, d_j] = [d_i, d_j] = 0,\\
&[h_i, e_{jl}] = a_{ij} e_{jl}, \qquad [h_i, f_{jl}] = -a_{ij} f_{jl},\\
&[d_i, e_{jl}] = \delta_{ij} e_{jl}, \qquad [d_i, f_{jl}] = -\delta_{ij} f_{jl},\\
&[e_{ik}, f_{jl}] = \delta_{ij}\delta_{kl} h_i,\\
&(ad\, e_{ik})^{1-a_{ij}}(e_{jl}) = (ad\, f_{ik})^{1-a_{ij}}(f_{jl}) = 0 \qquad \text{if } a_{ii} = 2;\ i \neq j,\\
&[e_{ik}, e_{jl}] = [f_{ik}, f_{jl}] = 0 \qquad \text{if } a_{ij} = 0
\end{aligned} \tag{1.1}$$

for $i, j \in I$, $k = 1, \ldots, m_i$, $l = 1, \ldots, m_j$.

The abelian subalgebra $\mathfrak{h} = (\bigoplus_{i \in I} \mathbf{F} h_i) \oplus (\bigoplus_{i \in I} \mathbf{F} d_i)$ is called the *Cartan subalgebra* of $\mathfrak{g}$. For each $j \in I$, define an $\mathbf{F}$-linear functional $\alpha_j \in \mathfrak{h}^*$ by

$$\alpha_j(h_i) = a_{ij}, \qquad \alpha_j(d_i) = \delta_{ij} \qquad \text{for } i, j \in I.$$

Let $\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathfrak{h}$. Note that $\Pi$ and $\Pi^\vee$ are linearly independent over $\mathbf{F}$. The elements of $\Pi$ (resp. $\Pi^\vee$) are called the *simple roots* (resp. *simple coroots*) of $\mathfrak{g}$.

Let $Q = \bigoplus_{i \in I} \mathbf{Z}\alpha_i$ be the free abelian group generated by $\alpha_i$'s ($i \in I$). We call $Q$ the *root lattice* of $\mathfrak{g}$. Set $Q_+ = \sum_{i \in I} \mathbf{Z}_{\geq 0}\alpha_i$, and $Q_- = -Q_+$. We define a partial ordering $\geq$ on $\mathfrak{h}^*$ by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. The generalized Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m})$ has the *root space decomposition* $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}\}$ is the *α-root space*. Note that $\mathfrak{g}_{\alpha_i} = \mathbf{C}e_{i,1} \oplus \cdots \oplus \mathbf{C}e_{i,m_i}$, and $\mathfrak{g}_{-\alpha_i} = \mathbf{C}f_{i,1} \oplus \cdots \oplus \mathbf{C}f_{i,m_i}$. We say that $\alpha \in Q$ is a *root* if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The number $\operatorname{mult}\alpha := \dim \mathfrak{g}_\alpha$ is called the *multiplicity* of the root $\alpha$. A root $\alpha > 0$ (resp. $\alpha < 0$) is called *positive* (resp. *negative*). We denote by $\Delta$, $\Delta^+$, and $\Delta^-$ the set of all roots, positive roots, and negative roots, respectively. Define the subspaces $\mathfrak{g}^{\pm} = \bigoplus_{\alpha \in \Delta^{\pm}} \mathfrak{g}_\alpha$. Then we have the *triangular decomposition*: $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$.

Since $A$ is symmetrizable, there is a symmetric bilinear form $(\,|\,)$ on $\mathfrak{h}^*$ satisfying $(\alpha_i \mid \alpha_j) = s_i a_{ij}$ for $i, j \in I$. We say that a root $\alpha$ is *real* if $(\alpha \mid \alpha) > 0$, and *imaginary* if $(\alpha \mid \alpha) \leq 0$. In particular, the simple root $\alpha_i$ is real if $a_{ii} = 2$, and imaginary if $a_{ii} \leq 0$. Note that the imaginary simple roots may have multiplicity $> 1$. For each $i \in I^{\mathrm{re}}$, let $r_i \in \mathrm{GL}(\mathfrak{h}^*)$ be the *simple reflection* on $\mathfrak{h}^*$ defined by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ for $\lambda \in \mathfrak{h}^*$. The subgroup $W$ of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the $r_i$'s ($i \in I^{\mathrm{re}}$) is called the *Weyl group* of $\mathfrak{g}$.

We denote by $U(\mathfrak{g})$ the *universal enveloping algebra* of $\mathfrak{g}$. Thus the algebra $U(\mathfrak{g})$ is the associative algebra over $\mathbf{F}$ with 1 generated by the elements $h_i$, $d_i$ ($i \in I$), $e_{ik}$, $f_{ik}$ ($i \in I$, $k = 1, \ldots, m_i$) with the defining relations:

$$
\begin{aligned}
&[h_i, h_j] = [h_i, d_j] = [d_i, d_j] = 0, \\
&h_i e_{jl} - e_{jl} h_i = a_{ij} e_{jl}, \qquad h_i f_{jl} - f_{jl} h_i = -a_{ij} f_{jl}, \\
&d_i e_{jl} - e_{jl} d_i = \delta_{ij} e_{jl}, \qquad d_i f_{jl} - f_{jl} d_i = -\delta_{ij} f_{jl}, \\
&e_{ik} f_{jl} - f_{jl} e_{ik} = \delta_{ij}\delta_{kl} h_i, \\
&\sum_{m+n=1-a_{ij}} (-1)^m \frac{e_{ik}^m}{m!} e_{jl} \frac{e_{ik}^n}{n!} = 0, \\
&\sum_{m+n=1-a_{ij}} (-1)^m \frac{f_{ik}^m}{m!} f_{jl} \frac{f_{ik}^n}{n!} = 0 \qquad \text{if } a_{ii} = 2;\ i \neq j, \\
&[e_{ik}, e_{jl}] = [f_{ik}, f_{jl}] = 0 \qquad \text{if } a_{ij} = 0
\end{aligned}
\tag{1.2}
$$

for $i, j \in I$, $k = 1, \ldots, m_i$, $l = 1, \ldots, m_j$.

We denote by $U(\mathfrak{h})$ the $\mathbf{F}$-subalgebra of $U(\mathfrak{g})$ with 1 generated by $h_i$, $d_i$ ($i \in I$), and $U(\mathfrak{g}^+)$ (resp. $U(\mathfrak{g}^-)$) the $\mathbf{F}$-subalgebra of $U(\mathfrak{g})$ with 1 generated by $e_{ik}$ (resp. $f_{ik}$) for $i \in I$, $k = 1, \dots, m_i$. The algebra $U(\mathfrak{g})$ has a Hopf algebra structure with comultiplication $\overline{\Delta}$, counit $\overline{\epsilon}$, and antipode $\overline{S}$ defined by

$$\begin{aligned} \overline{\Delta}(h_i) &= h_i \otimes 1 + 1 \otimes h_i, \\ \overline{\Delta}(d_i) &= d_i \otimes 1 + 1 \otimes d_i \qquad \text{for } i \in I, \\ \overline{\Delta}(e_{ik}) &= e_{ik} \otimes 1 + 1 \otimes e_{ik}, \\ \overline{\Delta}(f_{ik}) &= f_{ik} \otimes 1 + 1 \otimes f_{ik} \qquad \text{for } i \in I;\ k = 1, \dots, m_i, \end{aligned} \tag{1.3}$$

$$\begin{aligned} \overline{\varepsilon}(h_i) &= \overline{\varepsilon}(d_i) = 0 \qquad \text{for } i \in I, \\ \overline{\varepsilon}(e_{ik}) &= \overline{\varepsilon}(f_{ik}) = 0 \qquad \text{for } i \in I;\ k = 1, \dots, m_i, \end{aligned} \tag{1.4}$$

$$\begin{aligned} \overline{S}(h_i) &= -h_i, \qquad \overline{S}(d_i) = -d_i \qquad \text{for } i \in I, \\ \overline{S}(e_{ik}) &= -e_{ik}, \qquad \overline{S}(f_{ik}) = -f_{ik} \qquad \text{for } i \in I;\ k = 1, \dots, m_i. \end{aligned} \tag{1.5}$$

A $\mathfrak{g}$-module $V$ is called *diagonalizable* if it admits a *weight space decomposition* $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$, where $V_\mu = \{v \in V \mid h \cdot v = \mu(h)v$ for all $h \in \mathfrak{h}\}$. If $\dim_{\mathbf{F}} V_\mu < \infty$ for all $\mu \in \mathfrak{h}^*$, we define the *character* of $V$ to be

$$\operatorname{ch} V = \sum_{\mu \in \mathfrak{h}^*} (\dim_{\mathbf{F}} V_\mu) e^\mu,$$

where $e^\mu$ are the basis elements of the group algebra $\mathbf{F}[\mathfrak{h}^*]$ with the multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for $\mu$, $\nu \in \mathfrak{h}^*$.

A diagonalizable $\mathfrak{g}$-module $V$ is called a *highest weight module* with highest weight $\lambda \in \mathfrak{h}^*$ if there is a nonzero vector $v_\lambda \in V$ such that (i) $e_{ik} \cdot v_\lambda = 0$ for all $i \in I$, $k = 1, \dots, m_i$; (ii) $h \cdot v_\lambda = \lambda(h) v_\lambda$ for all $h \in \mathfrak{h}$; (iii) $V = U(\mathfrak{g}) \cdot v_\lambda$. The vector $v_\lambda$ is called a *highest weight vector*. For a highest weight module $V$ with highest weight $\lambda$, we have (i) $V = U(\mathfrak{g}^-) \cdot v_\lambda$; (ii) $V = \bigoplus_{\mu \le \lambda} V_\mu$, $V_\lambda = \mathbf{F} v_\lambda$; and (iii) $\dim_{\mathbf{F}} V_\mu < \infty$ for all $\mu \le \lambda$.

Let $\lambda \in \mathfrak{h}^*$ and consider the left deal $I(\lambda)$ of $U(\mathfrak{g})$ generated by the elements $e_{ik}$ ($i \in I$, $k = 1, \dots, m_i$) and $h - \lambda(h)1$ ($h \in \mathfrak{h}$). Let $M(\lambda) = U(\mathfrak{g})/I(\lambda)$, and define a $U(\mathfrak{g})$-module structure on $M(\lambda)$ by left multiplication. Then it is clear that $M(\lambda)$ is a highest weight module over $U(\mathfrak{g})$ with highest weight $\lambda$ and highest weight vector $v_\lambda = 1 + I(\lambda)$. The $U(\mathfrak{g})$-module $M(\lambda)$ is called the *Verma module* with highest weight $\lambda$.

PROPOSITION 1.2 [B2, K]. (a) *For every $\lambda \in \mathfrak{h}^*$, every highest weight module over $U(\mathfrak{g})$ with highest weight $\lambda$ is a homomorphic image of $M(\lambda)$.*

(b) *The Verma module $M(\lambda)$ is the unique module satisfying* (a) *up to isomorphism for every $\lambda \in \mathfrak{h}^*$.*

(c) *As a $U(\mathfrak{g}^-)$-module, $M(\lambda)$ is free of rank one generated by the highest weight vector $v_\lambda = 1 + I(\lambda)$.*

(d) *$M(\lambda)$ contains a unique maximal submodule $J(\lambda)$.*

For $\lambda \in \mathfrak{h}^*$, the unique irreducible quotient $V(\lambda) = M(\lambda)/J(\lambda)$ is called the *irreducible highest weight module* with highest weight $\lambda$.

For each $i \in I$, define $\mathbf{F}$-linear functionals $\Lambda_i$ and $\omega_i$ on $\mathfrak{h}$ by

$$\begin{aligned} &\Lambda_i(h_j) = \delta_{ij}, \qquad \Lambda_i(d_j) = 0, \\ &\omega_i(h_j) = 0, \qquad \omega_i(d_j) = \delta_{ij} \qquad \text{for all } j \in I, \end{aligned} \tag{1.6}$$

and let

$$P = \left( \oplus_{i \in I} \mathbf{Z}\Lambda_i \right) \oplus \left( \oplus_{i \in I} \mathbf{Z}\omega_i \right). \tag{1.7}$$

We call $P$ the *weight lattice* of $\mathfrak{g}$. On the other hand, the $\mathbf{Z}$-lattice

$$P^\vee = \left( \oplus_{i \in I} \mathbf{Z}h_i \right) \oplus \left( \oplus_{i \in I} \mathbf{Z}d_i \right) \tag{1.8}$$

is called the *dual weight lattice* of $\mathfrak{g}$. Let

$$P^+ = \{ \lambda \in P \mid \lambda(h_i) \geq 0 \text{ for all } i \in I \}.$$

The elements of $P^+$ are called the *dominant integral weights.*

PROPOSITION 1.3 [B2]. *If $V$ is a highest weight module over $U(\mathfrak{g})$ with highest weight $\lambda \in P^+$ and highest weight vector $v_\lambda$ such that*

(i) *if $\lambda(h_i) = 0$, then $f_{ik} \cdot v_\lambda = 0$ for $k = 1, \dots, m_i$,*

(ii) *if $a_{ii} = \rho$, then $f_{ik}^{\lambda(h_i)+1} \cdot v_\lambda = 0$,*

*then $V$ is isomorphic to the irreducible highest weight module $V(\lambda)$.*

Take an $\mathbf{F}$-linear functional $\rho \in \mathfrak{h}^*$ satisfying $\rho(h_i) = \frac{1}{2}a_{ii}$ for all $i \in I$, and let $T$ be the set of all imaginary simple roots counted with multiplicities. For $\lambda$, $\mu \in \mathfrak{h}^*$, we write $\lambda \perp \mu$ if $(\lambda \mid \mu) = 0$. Then the character of the irreducible highest weight module $V(\lambda)$ with highest weight $\lambda \in P^+$ is determined by the *Weyl–Kac–Borcherds formula.*

THEOREM 1.4 [B2, K].

$$\operatorname{ch} V(\lambda) = \frac{\sum\limits_{\substack{w \in W \\ F \subset T \\ F \perp \lambda}} (-1)^{l(w)+|F|} e^{w(\lambda+\rho-s(F))-\rho}}{\prod\limits_{\alpha \in \Delta^{-}} (1-e^{\alpha})^{\dim \mathfrak{g}_{\alpha}}}, \tag{1.10}$$

*where $F$ runs over all the finite subsets of $T$ such that any two elements of $F$ are mutually perpendicular. We denote by $|F|$ the number of elements in $F$ and $s(F)$ the sum of the elements in $F$.*

Letting $\lambda = 0$, we obtain the *denominator identity*:

$$\prod_{\alpha \in \Delta^{-}} (1-e^{\alpha})^{\dim \mathfrak{g}_{\alpha}} = \sum_{\substack{w \in W \\ F \subset T}} (-1)^{l(w)+|F|} e^{w(\rho-s(F))-\rho}. \tag{1.11}$$

Therefore the Weyl–Kac–Borcherds formula can be rephrased as

$$\operatorname{ch} V(\lambda) = \frac{\sum\limits_{\substack{w \in W \\ F \subset T \\ F \perp \lambda}} (-1)^{l(w)+|F|} e^{w(\lambda+\rho-s(F))-\rho}}{\sum\limits_{\substack{w \in W \\ F \subset T}} (-1)^{l(w)+|F|} e^{w(\rho-s(F))-\rho}}. \tag{1.12}$$

## 2. THE QUANTUM GROUPS $U_q(\mathfrak{g})$

In this section, we construct quantum groups $U_q(\mathfrak{g})$ associated with generalized Kac–Moody algebras with admissible Borcherds–Cartan matrices. A Borcherds–Cartan matrix $A = (a_{ij})_{i,j \in I}$ is said to be *admissible* if (i) $a_{ij} \in \mathbf{Z}$ for all $i, j \in I$; (ii) $a_{ii} \in 2\mathbf{Z} \setminus \{0\}$ for all $i \in I$; and (iii) there is a diagonal matrix $D = \operatorname{diag}(s_i \mid i \in I)$ with $s_i > 0$ $(i \in I)$ such that $DA$ is symmetric and $s_i a_{ii} \in \mathbf{Z} \setminus \{0\}$ for all $i \in I$. From now on, we assume that the Borcherds–Cartan matrix $A$ is admissible, and we let $t_i = a_{ii}/2$ for all $i \in I$. Note that an admissible Borcherds–Cartan matrix is symmetrizable. Thus there is a bilinear form $(\,|\,)$ on $\mathfrak{h}^*$ satisfying $(\alpha_i \mid \alpha_j) = s_i a_{ij}$ for $i, j \in I$. Recall that the dual weight lattice $P^{\vee}$ is given by

$$P^{\vee} = \left( \oplus_{i \in I} \mathbf{Z} h_i \right) \oplus \left( \oplus_{i \in I} \mathbf{Z} d_i \right).$$

DEFINITION 2.1. Let $\mathfrak{g} = \mathfrak{g}(A, \underline{m})$ be a generalized Kac–Moody algebra with an admissible Borcherds–Cartan matrix $A = (a_{ij})_{i,j \in I}$ of charge

$\underline{m} = (m_i \mid i \in I)$, and let $q$ be an indeterminate. The *quantum group* $U_q(\mathfrak{g})$ associated with $\mathfrak{g}$ is an associative algebra over $\mathbf{F}(q)$ with 1 generated by the elements $q^h$ $(h \in P^\vee)$, $e_{ik}$, $f_{ik}$ $(i \in I,\ k = 1, \dots, m_i)$ with the defining relations:

$$
\begin{aligned}
& q^0 = 1, \qquad q^h q^{h'} = q^{h+h'} \qquad \text{for } h, h' \in P^\vee, \\
& q^h e_{ik} q^{-h} = q^{\alpha_i(h)} e_{ik}, \qquad q^h f_{ik} q^{-h} = q^{-\alpha_i(h)} f_{ik} \qquad \text{for } h \in P^\vee, \\
& \quad i \in I;\ k = 1, \dots, m_i, \\
& e_{ik} f_{jl} - f_{jl} e_{ik} = \delta_{ij} \delta_{kl} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \qquad \text{where } q_i = q^{(\alpha_i | \alpha_i)}, \\
& K_i = \left(q^{h_i}\right)^{(\alpha_i | \alpha_i)} \qquad \text{for } i, j \in I;\ k = 1, \dots, m_i;\ l = 1, \dots, m_j, \\
& \sum_{m+n = 1 - a_{ij}} (-1)^m e_{ik}^{(m)} e_{jl} e_{ik}^{(n)} = 0, \\
& \sum_{m+n = 1 - a_{ij}} (-1)^m f_{ik}^{(m)} f_{jl} f_{ik}^{(n)} = 0 \qquad \text{if } a_{ii} = 2;\ i \neq j, \\
& \left[e_{ik}, e_{jl}\right] = \left[f_{ik}, f_{jl}\right] = 0 \qquad \text{if } a_{ij} = 0,
\end{aligned}
\tag{2.1}
$$

where $e_{ik}^{(n)} = e_{ik}^n/[n]_i!$, $f_{ik}^{(n)} = f_{ik}^n/[n]_i!$, $[n]_i! = \prod_{m=1}^n [m]_i$, $[m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1})$. The algebra $U_q(\mathfrak{g})$ has a Hopf algebra structure with comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$ defined by

$$
\begin{aligned}
& \Delta(q^h) = q^h \otimes q^h \qquad \text{for } h \in P^\vee, \\
& \Delta(e_{ik}) = e_{ik} \otimes K_i^{-1} + 1 \otimes e_{ik}, \\
& \Delta(f_{ik}) = f_{ik} \otimes 1 + K_i \otimes f_{ik} \qquad \text{for } i \in I;\ k = 1, \dots, m_i,
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
& \varepsilon(q^h) = 1 \qquad \text{for } h \in P^\vee, \\
& \varepsilon(e_{ik}) = \varepsilon(f_{ik}) = 0 \qquad \text{for } i \in I;\ k = 1, \dots, m_i,
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
& S(q^h) = q^{-h} \qquad \text{for } h \in P^\vee, \\
& S(e_{ik}) = -e_{iK} K_i, \qquad S(f_{ik}) = -K_i^{-1} f_{iK} \\
& \qquad \text{for } i \in I;\ k = 1, \dots, m_i.
\end{aligned}
\tag{2.4}
$$

We denote by $U_q(\mathfrak{h})$ the $\mathbf{F}(q)$-subalgebra of $U_q(\mathfrak{g})$ with 1 generated by the elements $q^h$ $(h \in P^\vee)$, and $U_q(\mathfrak{g}^+)$ (resp. $U_q(\mathfrak{g}^-)$) the $\mathbf{F}(q)$-subalgebra of $U_q(\mathfrak{g})$ with 1 generated by the elements $e_{ik}$ (resp. $f_{ik}$) for $i \in I$, $k = 1, \dots, m_i$. Thus the algebra $U_q(\mathfrak{h})$ is the group algebra $\mathbf{F}(q)[P^\vee]$ with a basis consisting of the elements $q^h$ $(h \in P^\vee)$. We will show that the quantum group $U_q(\mathfrak{g})$ has the *triangular decomposition*:

$$U_q(\mathfrak{g}) \cong U_q(\mathfrak{g}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{g}^+).$$

The idea of the proof is the same as that of [R], where the triangular decomposition was established for the quantum groups associated with the finite dimensional simple Lie algebras. We first prove the following.

LEMMA 2.2. *Let $U_q(\mathfrak{b}^+)$ (resp. $U_q(\mathfrak{b}^-)$) be the $\mathbf{F}(q)$-subalgebra of $U_q(\mathfrak{g})$ with 1 generated by the elements $q^h$ ($h \in P^\vee$) and $e_{ik}$ (resp. $f_{ik}$) for $i \in I$, $k = 1, \ldots, m_i$. Then we have*

$$U_q(\mathfrak{b}^+) \cong U_q(\mathfrak{h}) \otimes U_q(\mathfrak{g}^+), \tag{2.5}$$

$$U_q(\mathfrak{b}^-) \cong U_q(\mathfrak{g}^-) \otimes U_q(\mathfrak{h}). \tag{2.6}$$

*Proof.* We will prove (2.6) only; (2.5) can be proved in a similar way. Recall that $U_q(\mathfrak{h})$ has a basis $\{q^h \mid h \in P^\vee\}$ and let $\{e_\tau \mid \tau \in \Omega\}$ (resp. $\{f_\tau \mid \tau \in \Omega\}$) be a basis of $U_q(\mathfrak{g}^+)$ (resp. $U_q(\mathfrak{g}^-)$), where $e_\tau$ (resp. $f_\tau$) are monomials in $e_{ik}$ (resp. $f_{ik}$) indexed by a set $\Omega$. By the defining relations (2.1), the algebra $U_q(\mathfrak{b}^-)$ is spanned by the elements of the form $f_\tau q^h$ ($\tau \in \Omega$, $h \in P^\vee$). Thus we have a surjective $\mathbf{F}(q)$-linear map $U_q(\mathfrak{g}^-) \otimes U_q(\mathfrak{h}) \to U_q(\mathfrak{b}^-)$ defined by $f_\tau \otimes q^h \mapsto f_\tau q^h$. To show the injectivity of the above map, it suffices to show that the elements $f_\tau q^h$ ($\tau \in \Omega$, $h \in P^\vee$) are linearly independent over $\mathbf{F}(q)$.

Suppose $\sum_{\tau \in \Omega, h \in P^\vee} c_{\tau,h} f_\tau q^h = 0$ with $c_{\tau,h} \in \mathbf{F}(q)$ ($\tau \in \Omega$, $h \in P^\vee$). Then we have

$$\sum_{\beta \in Q_-} \left( \sum_{\substack{\deg f_\tau = \beta \\ h \in P^\vee}} c_{\tau,h} f_\tau q^h \right) = 0,$$

which yields $\sum_{\deg f_\tau = \beta} c_{\tau,h} f_\tau q^h = 0$ for all $\beta \in Q_-$. Let us write $\deg f_\tau = \beta = -\sum_{i \in I} k_i \alpha_i$ ($k_i \in \mathbf{Z}_{\geq 0}$). Since $f_\tau$ is a monomial in $f_{ik}$'s, by (2.2), we have

$$\Delta(f_\tau) = f_\tau \otimes 1 + (\text{intermediate terms}) + q^{h_\tau} \otimes f_\tau,$$

where $h_\tau = \sum_{i \in I} k_i(\alpha_i \mid \alpha_i) h_i$. It follows that

$$\begin{aligned} 0 &= \Delta\left( \sum_{\substack{\deg f_\tau = \beta \\ h \in P^\vee}} c_{\tau,h} f_\tau q^h \right) \\ &= \sum_{\substack{\deg g_\tau = \beta \\ h \in P^\vee}} c_{\tau,h}\big(f_\tau \otimes 1 + (\text{intermediate terms}) + q^{h_\tau} \otimes f_\tau\big)(q^h \otimes q^h) \\ &= \sum_{\substack{\deg f_\tau = \beta \\ h \in P^\vee}} c_{\tau,h}\big(f_\tau q^h \otimes q^h + (\text{intermediate terms}) + q^{h_\tau + h} \otimes f_\tau q^h\big). \end{aligned} \tag{2.7}$$

Since the terms of degree $(0, \beta)$ in (2.7) must be zero, we obtain

$$\sum_{\substack{\deg f_\tau = \beta \\ h \in P^\vee}} c_{\tau,h} q^{h_\tau + h} \otimes f_\tau q^h = 0.$$

Since $q^{h_\tau + h}$ $(h \in P^\vee)$ are linearly independent, we have $\sum_{\deg f_\tau = \beta} c_{\tau,h} \cdot f_\tau q^h = 0$ for all $h \in P^\vee$, which implies $\sum_{\deg f_\tau = \beta} c_{\tau,h} f_\tau = 0$. Therefore, $c_{\tau,h} = 0$ for all $\tau \in \Omega$, $h \in P^\vee$. ∎

We now show that the algebra $U_q(\mathfrak{g})$ has the triangular decomposition.

PROPOSITION 2.3. *There is an* $\mathbf{F}(q)$*-linear isomorphism*

$$U_q(\mathfrak{g}) \cong U_q(\mathfrak{g}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{g}^+). \tag{2.8}$$

*Proof.* As in Lemma 2.2, it suffices to show that the elements $f_\tau q^h e_\mu$ $(\tau, \mu \in \Omega,\ h \in P^\vee)$ are linearly independent over $\mathbf{F}(q)$. Suppose $\sum_{\tau, \mu \in \Omega, h \in P^\vee} c_{\tau,h,\mu} f_\tau q^h e_\mu = 0$ with $c_{\tau,h,\mu} \in \mathbf{F}(q)$. We may assume that

$$\sum_{\substack{h \in P^\vee \\ \deg f_\tau + \deg e_\mu = \gamma}} c_{\tau,h,\mu} f_\tau q^h e_\mu = 0 \tag{2.9}$$

for all $\gamma \in Q$.

Let us write $\deg e_\mu = \sum_{i \in I} m_i \alpha_i$ and $\deg f_\tau = -\sum_{i \in I} k_i \alpha_i$ $(m_i, k_i \in \mathbf{Z}_{\geq 0})$. Then, by (2.2), we get

$$\Delta(e_\mu) = e_\mu \otimes q^{-h_\mu} + (\text{intermediate terms}) + 1 \otimes e_\mu,$$

$$\Delta(f_\tau) = f_\tau \otimes 1 + (\text{intermediate terms}) + q^{h_\tau} \otimes f_\tau,$$

where $h_\mu = \sum_{i \in I} m_i (\alpha_i \mid \alpha_i) h_i$ and $h_\tau = \sum_{i \in I} k_i (\alpha_i \mid \alpha_i) h_i$. Thus we have

$$\begin{aligned} 0 &= \Delta\left( \sum_{\substack{h \in P^\vee \\ \deg f_\tau + \deg e_\mu = \gamma}} c_{\tau,h,\mu} f_\tau q^h e_\mu \right) \\ &= \sum_{\substack{h \in P^\vee \\ \deg f_\tau + \deg e_\mu = \gamma}} c_{\tau,h,\mu} \big( f_\tau \otimes 1 + (\text{intermediate terms}) + q^{h_\tau} \otimes f_\tau \big) \\ &\qquad \times (q^h \otimes q^h) \big( e_\mu \otimes q^{-h_\mu} + (\text{intermediate terms}) + 1 \otimes e_\mu \big). \end{aligned} \tag{2.10}$$

Consider the total ordering $\leq$ on $Q$ given by the height and lexicographical ordering, and let $\Omega_0$ (resp. $\Omega_1$) be the set of all $\tau \in \Omega$ (resp. $\mu \in \Omega$) such that $\deg f_\tau$ (resp. $\deg e_\mu$) is minimal (resp. maximal) among the terms in (2.9) with respect to $\leq$. Since $\deg f_\tau \in Q_-$, $\deg e_\mu \in Q_+$, and

$\deg f_\tau + \deg e_\mu = \gamma$, it is clear that $\tau \in \Omega_0$ if and only if $\mu \in \Omega_1$. Note that the terms of degree $(\alpha, \beta) = $ (maximal, minimal) in (2.10) must be zero. Hence we obtain

$$\sum_{\substack{h \in P^\vee \\ \tau \in \Omega_0 \\ \mu \in \Omega_1}} c_{\tau,h,\mu} q^{h_\tau + h} e_\mu \otimes f_\tau q^{h - h_\mu} = 0.$$

Since $f_\tau q^{h-h_\tau}$ ($\tau \in \Omega_0$, $h \in P^\vee$) are linearly independent by Lemma 2.2, we have

$$\sum_{\mu \in \Omega_1} c_{\tau,h,\mu} q^{h_\tau + h} e_\mu = 0,$$

which yields $c_{\tau,h,\mu} = 0$ for all $h \in P^\vee$, $\tau \in \Omega_0$, $\mu \in \Omega_1$.

Repeating the same argument, we conclude that $c_{\tau,h,\mu} = 0$ for all $h \in P^\vee$, $\tau$, $\mu \in \Omega$. ∎

A $U_q(\mathfrak{g})$-module $V^q$ is said to be *diagonalizable* if it admits a *weight space decomposition* $V^q = \bigoplus_{\mu \in P} V^q_\mu$, where $V^q_\mu = \{v \in V^q \mid q^h \cdot v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}$. If $\dim_{\mathbf{F}(q)} V^q_\mu < \infty$ for all $\mu \in P$, we define the *character* of $V^q$ to be

$$\operatorname{Ch} V^q = \sum_{\mu \in P} (\dim_{\mathbf{F}(q)} V^q_\mu) e^\mu,$$

where $e^\mu$ are the basis elements of the group algebra $\mathbf{F}(q)[P]$ with the multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for $\mu, \nu \in P$.

An element $v \in V^q_\mu$ is called a *primitive vector* of weight $\mu$ if $e_{ik} \cdot v = 0$ for all $i \in I$, $k = 1, \ldots, m_i$. A diagonalizable $U_q(\mathfrak{g})$-module $V^q$ is called a *highest weight module* with highest weight $\lambda \in P$ if there is a nonzero vector $v_\lambda \in V^q$ such that (i) $e_{ik} \cdot v_\lambda = 0$ for all $i \in I$, $k = 1, \ldots, m_i$; (ii) $q^h \cdot v_\lambda = q^{\lambda(h)} v_\lambda$ for all $h \in P^\vee$; (iii) $V^q = U_q(\mathfrak{g}) \cdot v_\lambda$. The vector $v_\lambda$ is called a *highest weight vector*. For a highest weight $U_q(\mathfrak{g})$-module $V^q$ with highest weight $\lambda \in P$, we have (i) $V^q = U_q(\mathfrak{g}^-) \cdot v_\lambda$; (ii) $V^q = \bigoplus_{\mu \le \lambda} V^q_\mu$, $V^q_\lambda = \mathbf{F}(q) v_\lambda$; and (iii) $\dim_{\mathbf{F}(q)} V^q_\mu < \infty$ for all $\mu \le \lambda$, where $U_q(\mathfrak{g}^-)$ denotes the $\mathbf{F}(q)$-subalgebra of $U_q(\mathfrak{g})$ with 1 generated by the elements $f_{ik}$ ($i \in I$, $k = 1, \ldots, m_i$).

Let $\lambda \in P$ and consider the left ideal $I_q(\lambda)$ of $U_q(\mathfrak{g})$ generated by the elements $e_{ik}$ ($i \in I$, $k = 1, \ldots, m_i$) and $q^h - q^{\lambda(h)} 1$ ($h \in P^\vee$). Let $M^q(\lambda) = U_q(\mathfrak{g}) / I_q(\lambda)$, and define a $U_q(\mathfrak{g})$-module structure on $M^q(\lambda)$ by left multiplication. Then it is clear that $M^q(\lambda)$ is a highest weight module over $U_q(\mathfrak{g})$ with highest weight $\lambda$ and highest weight vector $v_\lambda = 1 + I_q(\lambda)$. The $U_q(\mathfrak{g})$-module $M^q(\lambda)$ is called the *Verma module* with highest weight $\lambda$.

PROPOSITION 2.4 (cf. [K, Proposition 9.2]). (a) *For every $\lambda \in P$, every highest weight module over $U_q(\mathfrak{g})$ with highest weight $\lambda$ is a homomorphic image of $M^q(\lambda)$.*

(b) *The Verma module $M^q(\lambda)$ is the unique module satisfying* (a) *up to isomorphism for very $\lambda \in P$.*

(c) *As a $U_q(\mathfrak{g}^-)$-module, $M^q(\lambda)$ is free of rank one generated by the highest weight vector $v_\lambda = 1 + I_q(\lambda)$.*

(d) *$M^q(\lambda)$ contains a unique maximal submodule $J^q(\lambda)$.*

*Proof.* (a) If $V^q$ is a highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda$ and highest weight vector $w_\lambda$, then the map $M^q(\lambda) \to V^q$ given by $u \cdot (1 + I_q(\lambda)) \mapsto u \cdot w_\lambda$ defines a surjective $U_q(\mathfrak{g})$-module homomorphism.

(b) If $M_1^q(\lambda)$ and $M_2^q(\lambda)$ are two Verma modules with highest weight $\lambda$, then by (a) there is a surjective $U_q(\mathfrak{g})$-module homomorphism $\phi$: $M_1^q(\lambda) \to M_2^q(\lambda)$. In particular, $\phi(M_1^q(\lambda)_\mu) = M_2^q(\lambda)_\mu$ for all $\mu \in P$, and hence $\dim_{\mathbf{F}(q)} M_1^q(\lambda)_\mu \geq \dim_{\mathbf{F}(q)} M_2^q(\lambda)_\mu$ for all $\mu \in P$. Conversely, by (a) again, there is a surjective $U_q(\mathfrak{g})$-module homomorphism $\psi : M_2^q(\lambda) \to M_1^q(\lambda)$ showing that $\dim_{\mathbf{F}(q)} M_2^q(\lambda)_\mu \geq \dim_{\mathbf{F}(q)} M_1^q(\lambda)_\mu$ for all $\mu \in P$. Hence $\dim_{\mathbf{F}(q)} M_1^q(\lambda)_\mu = \dim_{\mathbf{F}(q)} M_2^q(\lambda)_\mu$ for all $\mu \in P$, and therefore $\phi$ and $\psi$ are $U_q(\mathfrak{g})$-module isomorphisms.

(c) Since every element $u$ of $U_q(\mathfrak{g})$ can be written as a sum of elements of the form $u^- u^0 u^+$, where $u^\pm \in U_q(\mathfrak{g}^\pm)$ and $u^0 \in U_q(\mathfrak{h})$, every element of $M^q(\lambda)$ has the form $u^- \cdot (1 + I_q(\lambda))$ for some $u^- \in U_q(\mathfrak{g}^-)$. If $u^- \cdot (1 + I_q(\lambda)) = 0$, then $u^- \in I_q(\lambda)$, which is generated by $e_{ik}$ ($i \in I$; $k = 1, \ldots, m_i$) and $q^h - q^{\lambda(h)} 1$ ($h \in P^\vee$). Hence $u^-$ must be zero, and our assertion follows.

(d) Note that any proper submodule $W^q$ of $M^q(\lambda)$ does not contain the highest weight vector $1 + I_q(\lambda)$, for otherwise we would have $W^q = M^q(\lambda)$. Thus the sum of proper submodules is again a proper submodule of $M^q(\lambda)$. Let $J^q(\lambda)$ be the sum of all the proper submodules of $M^q(\lambda)$. Then $J^q(\lambda)$ is the unique maximal submodule of $M^q(\lambda)$. ∎

For $\lambda \in P$, the unique irreducible quotient $V^q(\lambda) = M^q(\lambda)/J^q(\lambda)$ is called the *irreducible highest weight module* over $U_q(\mathfrak{g})$ with highest weight $\lambda$.

## 3. A-FORMS

Let $A = (a_{ij})_{i,j \in I}$ be an admissible Borcherds–Cartan matrix with charge $\underline{m} = (m_i \mid i \in I)$, and let $U_q(\mathfrak{g})$ be the quantum group associated with the generalized Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m})$. Recall that $q_i = q^{(\alpha_i|\alpha_i)}$ and $K_i = (q^{h_i})^{(\alpha_i|\alpha_i)}$ for $i \in I$. We also define $D_i = (q^{d_i})^{(\alpha_i|\alpha_i)}$ $(i \in I)$. Then for $i, j \in I$, $k = 1, \ldots, m_j$, we have

$$\begin{aligned} K_i e_{jk} K_i^{-1} &= q_i^{a_{ij}} e_{jk}, & K_i f_{jk} K_i^{-1} &= q_i^{-a_{ij}} f_{jk}, \\ D_i e_{jk} D_i^{-1} &= q_i^{\delta_{ij}} e_{jk}, & D_i f_{jk} D_i^{-1} &= q_i^{-\delta_{ij}} f_{jk}. \end{aligned} \tag{3.1}$$

For $i \in I$, $c \in \mathbf{Z}$, $N \in \mathbf{Z}_{\geq 0}$, we define

$$\begin{bmatrix} K_i; c \\ N \end{bmatrix} = \prod_{r=1}^{N} \frac{K_i q_i^{c-r+1} - K_i^{-1} q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}}, \tag{3.2}$$

$$\begin{bmatrix} D_i; c \\ N \end{bmatrix} = \prod_{r=1}^{N} \frac{D_i q_i^{c-r+1} - D_i^{-1} q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}}, \tag{3.3}$$

For $i \in I$ and $N \in \mathbf{Z}_{>0}$, let

$$\begin{aligned} \{N\}_i &= \frac{q_i^{t_i N} - q_i^{-t_i N}}{q_i^{t_i} - q_i^{-t_i}} \\ &= q_i^{t_i(N-1)} + q_i^{t_i(N-3)} + \cdots + q_i^{-t_i(N-3)} + q_i^{-t_i(N-1)}, \end{aligned} \tag{3.4}$$

and define $e_{ik}^{\{N\}} = e_{ik}^N/\{N\}_i!$, $f_{ik}^{\{N\}} = f_{ik}^N/\{N\}_i!$, where $\{N\}_i! = \prod_{n=1}^N \{n\}_i$. Note that if $a_{ii} = 2$, then $t_i = 1$, and, hence, $\{N\}_i = [N]_i$, $e_{ik}^{\{N\}} = e_{ik}^{(N)}$, $f_{ik}^{\{N\}} = f_{ik}^{(N)}$. For $i \in I$, $c$, $M \in \mathbf{Z}$, and $N \in \mathbf{Z}_{>0}$, we define

$$\begin{Bmatrix} K_i; c; M \\ N \end{Bmatrix} = \frac{\prod_{r=1}^N K_i q_i^{t_i(c-r+1)+M} - K_i^{-1} q_i^{-t_i(c-r+1)-M}}{\{N\}_i!(q_i - q_i^{-1})^N}. \tag{3.5}$$

Let $\mathbf{A} = \mathbf{F}[q, q^{-1}, (1/\{N\}_i)\,(i \in I, N \in \mathbf{Z}_{>0})]$, and we define the $\mathbf{A}$-*form* $U_{\mathbf{A}}$ of the quantum group $U_q(\mathfrak{g})$ to be the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g})$ with 1 generated by the elements $e_{ik}^{\{N\}}$, $f_{ik}^{\{N\}}$ $(i \in I,\ k = 1, \ldots, m_i,\ N \in \mathbf{Z}_{>0})$, $q^h$ $(h \in P^\vee)$, $\begin{bmatrix} K_i; c \\ N \end{bmatrix}$, $\begin{bmatrix} D_i; c \\ N \end{bmatrix}$, $\begin{Bmatrix} K_i; c; M \\ N \end{Bmatrix}$ $(i \in I,\ c,\ M \in \mathbf{Z},\ N \in \mathbf{Z}_{\geq 0})$. We denote by $U_{\mathbf{A}}^+$ (resp. $U_{\mathbf{A}}^-$) the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g})$ with 1 generated by $e_{ik}^{\{N\}}$ (resp. $f_{ik}^{\{N\}}$) for $i \in I$, $k = 1, \ldots, m_i$, $N \in \mathbf{Z}_{>0}$, and $U_{\mathbf{A}}^0$ the $\mathbf{A}$-

subalgebra of $U_q(\mathfrak{g})$ with 1 generated by $q^h$ ($h \in P^\vee$), $\begin{bmatrix} K_i; c \\ N \end{bmatrix}$, $\begin{bmatrix} D_i; c \\ N \end{bmatrix}$, $\begin{Bmatrix} K_i; c; M \\ N \end{Bmatrix}$ ($i \in I$, $c$, $M \in \mathbf{Z}$, $N \in \mathbf{Z}_{\geq 0}$). We have the following commutation relations for the generators of $U_{\mathbf{A}}$.

LEMMA 3.1. *For $i, j \in I$, $k = 1, \ldots, m_i$, $l = 1, \ldots, m_j$, $c$, $M \in \mathbf{Z}$, and $s$, $N$, $N' \in \mathbf{Z}_{\geq 0}$, we have*

$$e_{jl}^{\{N\}} \begin{bmatrix} K_i; c \\ s \end{bmatrix} = \begin{bmatrix} K_i; c - Na_{ij} \\ s \end{bmatrix} e_{jl}^{\{N\}}, \tag{3.6}$$

$$\begin{bmatrix} K_i; c \\ s \end{bmatrix} f_{jl}^{\{N\}} = f_{jl}^{\{N\}} \begin{bmatrix} K_i; c - Na_{ij} \\ s \end{bmatrix}, \tag{3.7}$$

$$e_{jl}^{\{N\}} \begin{bmatrix} D_i; c \\ s \end{bmatrix} = \begin{bmatrix} D_i; c - N\delta_{ij} \\ s \end{bmatrix} e_{jl}^{\{N\}}, \tag{3.8}$$

$$\begin{bmatrix} D_i; c \\ s \end{bmatrix} f_{jl}^{\{N\}} = f_{jl}^{\{N\}} \begin{bmatrix} D_i; c - N\delta_{ij} \\ s \end{bmatrix}, \tag{3.9}$$

$$e_{jl}^{\{N\}} \begin{Bmatrix} K_i; c; M \\ s \end{Bmatrix} = \begin{Bmatrix} K_i; c; M - Na_{ij} \\ s \end{Bmatrix} e_{jl}^{\{N\}}, \tag{3.10}$$

$$\begin{Bmatrix} K_i; c; M \\ s \end{Bmatrix} f_{jl}^{\{N\}} = f_{jl}^{\{N\}} \begin{Bmatrix} K_i; c; M - Na_{ij} \\ s \end{Bmatrix}, \tag{3.11}$$

$$e_{ik}^{\{N\}} f_{jl}^{\{N'\}} = f_{jl}^{\{N'\}} e_{ik}^{\{N\}} \qquad \text{if } i \neq j \text{ or } k \neq l, \tag{3.12}$$

$$e_{ik}^{\{N\}} f_{ik}^{\{N'\}} = \sum_{s \geq 0} f_{ik}^{\{N'-s\}} \begin{Bmatrix} K_i; 2s - N - N'; 0 \\ s \end{Bmatrix} e_{ik}^{\{N-s\}}. \tag{3.13}$$

*Proof.* Equations (3.6)–(3.11) are immediate consequences of (3.1)–(3.5), and (3.12), (3.13) are proved by induction. ∎

As an immediate consequence of Lemma 3.1, we have the *triangular decomposition* of the algebra $U_{\mathbf{A}}$:

$$U_{\mathbf{A}} \cong U_{\mathbf{A}}^- \otimes U_{\mathbf{A}}^0 \otimes U_{\mathbf{A}}^+. \tag{3.14}$$

In particular, every element $u$ of $U_{\mathbf{A}}$ can be written as a sum of monomials of the form $u^- u^0 u^+$, where $u^0 \in U_{\mathbf{A}}^0$, $u^\pm \in U_{\mathbf{A}}^\pm$.

COROLLARY 3.2. *Let $V^q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P^+$ and highest weight vector $v_\lambda$.*

(a) *If $\lambda(h_i) = 0$, then $f_{ik} \cdot v_\lambda = 0$ for $k = 1, \dots, m_i$.*

(b) *If $a_{ii} = 2$, then $f_{ik}^{\lambda(h_i)+1} \cdot v_\lambda = 0$.*

*Proof.* (a) If $k \neq l$, then $e_{il} f_{ik} \cdot v_\lambda = f_{ik} e_{il} \cdot v_\lambda = 0$. Note that $e_{ik} f_{ik} = f_{ik} e_{ik} + (K_i - K_i^{-1})/(q_i - q_i^{-1})$ and $e_{ik} \cdot v_\lambda = 0$. Thus if $\lambda(h_i) = 0$, then we have

$$\begin{aligned} e_{ik} f_{ik} \cdot v_\lambda &= f_{ik} e_{ik} \cdot v_\lambda + \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} v_\lambda \\ &= \frac{q_i^{\lambda(h_i)} - q_i^{-\lambda(h_i)}}{q_i - q_i^{-1}} v_\lambda = 0. \end{aligned}$$

For $j \neq i$, $e_{jl} f_{ik} \cdot v_\lambda = f_{ik} e_{jl} \cdot v_\lambda = 0$ for $l = 1, \dots, m_j$. Hence $f_{ik} \cdot v_\lambda$ is a primitive vector in $V^q(\lambda)$. Since $V^q(\lambda)$ is irreducible, we must have $f_{ik} \cdot v_\lambda = 0$, for otherwise $f_{ik} \cdot v_\lambda$ would generate a submodule of $V^q(\lambda)$ with highest weight $\lambda - \alpha_i \neq \lambda$, which is a contradiction.

(b) By (3.12) and (3.13), for $N \in \mathbf{Z}_{>0}$, we have

$$e_{ik} f_{ik}^{\{N\}} = f_{ik}^{\{N\}} e_{ik} + f_{ik}^{\{N-1\}} \frac{K_i q_i^{-t_i(N-1)} - K_i^{-1} q_i^{t_i(N-1)}}{q_i - q_i^{-1}}.$$

If $a_{ii} = 2$, then $t_i = 1$ and

$$\begin{aligned} &e_{ik} f_{ik}^{\lambda(h_i)+1} \cdot v_\lambda \\ &\quad = \{\lambda(h_i) + 1\}_i! e_{ik} f_{ik}^{\{\lambda(h_i)+1\}} \cdot v_\lambda \\ &\quad = \{\lambda(h_i) + 1\}_i! \left( f_{ik}^{\{\lambda(h_i)+1\}} e_{ik} \cdot v_\lambda + f_{ik}^{\{\lambda(h_i)\}} \frac{K_i q_i^{-\lambda(h_i)} - K_i^{-1} q_i^{\lambda(h_i)}}{q_i - q_i^{-1}} v_\lambda \right) \\ &\quad = \{\lambda(h_i) + 1\}_i! f_{ik}^{\{\lambda(h_i)\}} \frac{q_i^{\lambda(h_i) - \lambda(h_i)} - q_i^{-\lambda(h_i) + \lambda(h_i)}}{q_i - q_i^{-1}} v_\lambda = 0. \end{aligned}$$

For $j \neq i$, $e_{jl} f_{ik}^{\lambda(h_i)+1} \cdot v_\lambda = f_{ik}^{\lambda(h_i)+1} e_{jl} \cdot v_\lambda = 0$. Therefore, $f_{ik}^{\lambda(h_i)+1} \cdot v_\lambda$ is a primitive vector of weight $\lambda - (\lambda(h_i) + 1)\alpha_i \neq \lambda$, and hence $f_{ik}^{\lambda(h_i)+1} \cdot v_\lambda = 0$. ∎

Let $\lambda \in P$, and let $V^q$ be a highest weight module over $U_q(\mathfrak{g})$ with highest weight $\lambda$ and highest weight vector $v_\lambda$. We define the $\mathbf{A}$-*form* $V^{\mathbf{A}}$ of $V^q$ to be the $U_{\mathbf{A}}$-submodule of $V^q$ generated by $v_\lambda$. That is, $V^{\mathbf{A}} = U_{\mathbf{A}} \cdot v_\lambda$.

PROPOSITION 3.3. $V^{\mathbf{A}} = U_{\mathbf{A}}^{-} \cdot v_\lambda$.

*Proof.* Recall that every element $u$ of $U_{\mathbf{A}}$ can be written as a sum of monomials of the form $u^{-}u^{0}u^{+}$, where $u^0 \in U_{\mathbf{A}}^0$, $u^{\pm} \in U_{\mathbf{A}}^{\pm}$. By definition, $u^{+} \cdot v_\lambda = 0$ unless $u^{+} \in \mathbb{A}$, and $q^h \cdot v_\lambda = q^{\lambda(h)} v_\lambda \in \mathbf{A} v_\lambda$. For $i \in I$, $c \in \mathbf{Z}$, and $N \in \mathbf{Z}_{\geq 0}$, we have

$$\begin{bmatrix} K_i; c \\ N \end{bmatrix} \cdot v_\lambda = \prod_{r=1}^{N} \frac{q_i^{\lambda(h_i)+c-r+1} - q_i^{-(\lambda(h_i)+c-r+1)}}{q_i^r - q_i^{-r}} v_\lambda .$$

Note that

$$\prod_{r=1}^{N} \frac{q_i^{\lambda(h_i)+c-r+1} - q_i^{-(\lambda(h_i)+c-r+1)}}{q_i^r - q_i^{-r}} = \frac{[\lambda(h_i)+c]_i!}{[N]_i![\lambda(h_i)+c-N]_i!} \in \mathbf{A}.$$

Hence $\begin{bmatrix} K_i; c \\ N \end{bmatrix} \cdot v_\lambda \in \mathbf{A} v_\lambda$ and, similarly, $\begin{bmatrix} D_i; c \\ N \end{bmatrix} \cdot v_\lambda \in \mathbf{A} v_\lambda$.

Finally, for $i \in I$, $c$, $M \in \mathbf{Z}$, and $N \in \mathbf{Z}_{\geq 0}$, we have

$$\begin{Bmatrix} K_i; c; M \\ N \end{Bmatrix} \cdot v_\lambda = \frac{\prod_{r=1}^{N} \left( q_i^{t_i(c-r+1)+M+\lambda(h_i)} - q_i^{-(t_i(c-r+1)+M+\lambda(h_i))} \right)}{\{N\}_i!(q_i - q_i^{-1})^N} v_\lambda .$$

Since

$$\frac{q_i^{t_i(c-r+1)+M+\lambda(h_i)} - q_i^{-(t_i(c-r+1)+M+\lambda(h_i))}}{q_i - q_i^{-1}} \in \mathbf{F}[q, q^{-1}]$$

for all $r = 1, \dots, N$,

$$\frac{\prod_{r=1}^{N} \left( q_i^{t_i(c-r+1)+M+\lambda(h_i)} - q_i^{-(t_i(c-r+1)+M+\lambda(h_i))} \right)}{\{N\}_i!(q_i - q_i^{-1})^N} \in \mathbf{A},$$

which implies $\begin{Bmatrix} K_i; c; M \\ N \end{Bmatrix} \cdot v_\lambda \in \mathbf{A} v_\lambda$. Therefore, $u^{-}u^{0}u^{+} \cdot v_\lambda \in \mathbf{A} u^{-} \cdot v_\lambda \subset U_{\mathbf{A}}^{-} \cdot v_\lambda$. It follows that $V^{\mathbf{A}} = U_{\mathbf{A}}^{-} \cdot v_\lambda$. ∎

PROPOSITION 3.4. *The map $\phi : \mathbf{F}(q) \otimes_{\mathbf{A}} V^{\mathbf{A}} \to V^q$ given by $f \otimes v \mapsto fv$ ($f \in \mathbf{F}(q)$, $v \in V^{\mathbf{A}}$) is an $\mathbf{F}(q)$-linear isomorphism.*

*Proof.* By Proposition 3.3, the $\mathbf{A}$-form $V^{\mathbf{A}}$ of $V^q$ is spanned over $\mathbf{A}$ by the monomials of the form $f_{i_1,k_1}^{\{N_1\}} \cdots f_{i_r,k_r}^{\{N_r\}} \cdot v_\lambda$ and $V^q$ is spanned over $\mathbf{F}(q)$

by the monomials of the form $f_{i_1,k_1}^{N_{i_1}} \cdots f_{i_r,k_r}^{N_{i_r}} \cdot v_\lambda$ for $i_j \in I$, $k_j = 1, \dots, m_{i_j}$, $N_{i_j} \in \mathbf{Z}_{\geq 0}$. Then $\phi$ maps $f \otimes f_{i_1,k_1}^{\{N_{i_1}\}} \cdots f_{i_r,k_r}^{\{N_{i_r}\}} \cdot v_\lambda$ ($f \in \mathbf{F}(q)$) to

$$\frac{f}{\{N_{i_1}\}_{i_1}! \cdots \{N_{i_r}\}_{i_r}!} f_{i_1,k_1}^{N_{i_1}} \cdots f_{i_r,k_r}^{N_{i_r}} \cdot v_\lambda.$$

Define an $\mathbf{F}(q)$-linear map $\psi : V^q \to \mathbf{F}(q) \otimes_{\mathbf{A}} V^{\mathbf{A}}$ by

$$\psi\left(f_{i_1,k_1}^{N_{i_1}} \cdots f_{i_r,k_r}^{N_{i_r}} \cdot v_\lambda\right) = \{N_{i_1}\}_{i_1}! \cdots \{N_{i_r}\}_{i_r}! \otimes f_{i_1,k_1}^{\{N_{i_1}\}} \cdots f_{i_r,k_r}^{\{N_{i_r}\}} \cdot v_\lambda.$$

If $f_{i_1,k_1}^{N_{i_1}} \cdots f_{i_r,k_r}^{N_{i_r}} \cdot v_\lambda = f_{j_1,l_1}^{M_{j_1}} \cdots f_{j_s,l_s}^{M_{j_s}} \cdot v_\lambda$, then, since both $\{N_{i_1}\}_{i_1}! \cdots \{N_{i_r}\}_{i_r}! \in \mathbf{A}$ and $\{M_{j_1}\}_{j_1}! \cdots \{M_{j_s}\}_{j_s}! \in \mathbf{A}$, we have

$$\begin{aligned}
&\{N_{i_1}\}_{i_1}! \cdots \{N_{i_r}\}_{i_r}! f_{i_1,k_1}^{\{N_{i_1}\}} \cdots f_{i_r,k_r}^{\{N_{i_r}\}} \cdot v_\lambda \\
&\quad = f_{i_1,k_1}^{N_{i_1}} \cdots f_{i_r,k_r}^{N_{i_r}} \cdot v_\lambda \\
&\quad = f_{j_1,l_1}^{M_{j_1}} \cdots f_{j_s,l_s}^{M_{j_s}} \cdot v_\lambda = \{M_{j_1}\}_{j_1}! \cdots \{M_{j_s}\}_{j_s}! f_{j_1,l_1}^{\{M_{j_1}\}} \cdots f_{j_s,l_s}^{\{M_{j_s}\}} \cdot v_\lambda
\end{aligned}$$

in $U_{\mathbf{A}}$. It follows that

$$\begin{aligned}
&\{N_{i_1}\}_{i_1}! \cdots \{N_{i_r}\}_{i_r}! \otimes f_{i_1,k_1}^{\{N_{i_1}\}} \cdots f_{i_r,k_r}^{\{N_{i_r}\}} \cdot v_\lambda \\
&\quad = 1 \otimes \{N_{i_1}\}_{i_1}! \cdots \{N_{i_r}\}_{i_r}! f_{i_1,k_1}^{\{N_{i_1}\}} \cdots f_{i_r,k_r}^{\{N_{i_r}\}} \cdot v_\lambda \\
&\quad = 1 \otimes \{M_{j_1}\}_{j_1}! \cdots \{M_{j_s}\}_{j_s}! f_{j_1,l_1}^{\{M_{j_1}\}} \cdots f_{j_s,l_s}^{\{M_{j_s}\}} \cdot v_\lambda \\
&\quad = \{M_{j_1}\}_{j_1}! \cdots \{M_{j_s}\}_{j_s}! \otimes f_{j_1,l_1}^{\{M_{j_1}\}} \cdots f_{j_s,l_s}^{\{M_{j_s}\}} \cdot v_\lambda.
\end{aligned}$$

Therefore, the map $\psi$ is well defined. It is easy to see that $\phi$ and $\psi$ are inverses to each other, which proves our assertion. ∎

PROPOSITION 3.5. *For $\mu \in P$, define $V_\mu^{\mathbf{A}} = V^{\mathbf{A}} \cap V_\mu^q$. Then we have the weight space decomposition*: $V^{\mathbf{A}} = \bigoplus_{\mu \in P} V_\mu^{\mathbf{A}}$.

*Proof.* Let $v = v_1 + \cdots + v_p \in V^{\mathbf{A}}$, where $v_j \in V_{\mu_j}^q$ ($\mu_j \in P$, $j = 1, \dots, p$). We would like to show $v_j \in V^{\mathbf{A}}$ for all $j = 1, \dots, p$. We will show $v_1 \in V^{\mathbf{A}}$. The other cases can be proved in a similar way.

For $j = 1, 2, \dots, p$ and $i \in I$, write $\mu_j(h_i) = S_{ij}$ and $\mu_j(d_i) = T_{ij}$. For $j = 2, \dots, p$, since $\mu_j \neq \mu_1$, we can choose an index $i_j \in I$ such that $S_{i_j,j} \neq S_{i_j,1}$ or $T_{i_j,j} \neq T_{i_j,1}$. Let $I_0 = \{i_2, i_3, \dots, i_p\}$, and let $s$ be a positive integer such that $s \geq |S_{ij} - S_{i1}|$ and $s \geq |T_{ij} - T_{i1}|$ for all $i \in I_0$, $j =$

$1, \ldots, p$. (We can simply take the maximum of the numbers $|S_{ij} - S_{i1}|$, $|T_{ij} - T_{i1}|$ for $i \in I_0$, $j = 1, \ldots, p$.) We define an element $u$ of $U_{\mathbf{A}}$ to be

$$u = \prod_{i \in I_0} \begin{bmatrix} K_i; -S_{i1} + s \\ s \end{bmatrix} \begin{bmatrix} K_i; -S_{i1} - 1 \\ s \end{bmatrix} \times \begin{bmatrix} D_i; -T_{i1} + s \\ s \end{bmatrix} \begin{bmatrix} D_i; -T_{i1} - 1 \\ s \end{bmatrix}. \tag{3.14}$$

Then we have

$$\begin{bmatrix} D_i; -T_{i1} - 1 \\ s \end{bmatrix} \cdot v_1 = \prod_{r=1}^{s} \frac{D_i q_i^{-T_{i1} - r} - D_i^{-1} q_i^{T_{i1} + r}}{q_i^r - q_i^{-r}} v_1 = \prod_{r=1}^{s} \frac{q_i^{-r} - q_i^r}{q_i^r - q_i^{-r}} v_1 = (-1)^s v_1$$

and

$$\begin{bmatrix} D_i; -T_{i1} + s \\ s \end{bmatrix} \cdot v_1 = \prod_{r=1}^{s} \frac{D_i q_i^{-T_{i1} + s - r + 1} - D_i^{-1} q_i^{T_{i1} - s + r - 1}}{q_i^r - q_i^{-r}} v_1 = \prod_{r=1}^{s} \frac{q_i^{s - r + 1} - q_i^{-s + r - 1}}{q_i^r - q_i^{-r}} v_1 = v_1.$$

Similarly,

$$\begin{bmatrix} K_i; -S_{i1} - 1 \\ s \end{bmatrix} \cdot v_1 = (-1)^s v_1$$

and

$$\begin{bmatrix} K_i; -S_{i1} + s \\ s \end{bmatrix} \cdot v_1 = v_1.$$

Therefore, $u \cdot v_1 = (-1)^{2s(p-1)} v_1 = v_1$.

Let $j \neq 1$, say, $j = 2$. Then

$$\begin{bmatrix} D_i; -T_{i1} - 1 \\ s \end{bmatrix} \cdot v_2 = \prod_{r=1}^{s} \frac{q_i^{T_{i2} - T_{i1} - r} - q_i^{-(T_{i2} - T_{i1} - r)}}{q_i^r - q_i^{-r}} v_2$$

and

$$\begin{bmatrix} D_i; -T_{i1} + s \\ s \end{bmatrix} \cdot v_2 = \prod_{r=1}^{s} \frac{q_i^{T_{i2} - T_{i1} + s - r + 1} - q_i^{-(T_{i2} - T_{i1} + s - r + 1)}}{q_i^r - q_i^{-r}} v_2.$$

Thus

$$\prod_{i\in I_0}\begin{bmatrix} D_i; -T_{i1}+s \\ s\end{bmatrix}\begin{bmatrix} D_i; -T_{i1}-1 \\ s\end{bmatrix}\cdot v_2$$

$$= \prod_{i\in I_0}\prod_{r,t=1}^{s} \frac{\left(q_i^{T_{i2}-T_{i1}-r} - q_i^{-(T_{i2}-T_{i1}-r)}\right)\times\left(q_i^{T_{i2}-T_{i1}+s-t+1} - q_i^{-(T_{i2}-T_{i1}+s-t+1)}\right)}{\left(q_i^r - q_i^{-r}\right)\left(q_i^t - q_i^{-t}\right)} v_2.$$

Similarly, we have

$$\prod_{i\in I_0}\begin{bmatrix} K_i; -S_{i1}+s \\ s\end{bmatrix}\begin{bmatrix} K_i; -S_{i1}-1 \\ s\end{bmatrix}\cdot v_2$$

$$= \prod_{i\in I_0}\prod_{r,t=1}^{s} \frac{\left(q_i^{S_{i2}-S_{i1}-r} - q_i^{-(S_{i2}-S_{i1}-r)}\right)\times\left(q_i^{S_{i2}-S_{i1}+s-t+1} - q_i^{-(S_{i2}-S_{i1}+s-t+1)}\right)}{\left(q_i^r - q_i^{-r}\right)\left(q_i^t - q_i^{-t}\right)} v_2.$$

Consider the terms where $r + t = s + 1$:

$$\left(q_i^{T_{i2}-T_{i1}-r} - q_i^{-(T_{i2}-T_{i1}-r)}\right)\left(q_i^{T_{i2}-T_{i1}+s-t+1} - q_i^{-(T_{i2}-T_{i1}+s-t+1)}\right)$$
$$= q_i^{2(T_{i2}-T_{i1})} - q_i^{2r} - q_i^{-2r} + q_i^{-2(T_{i2}-T_{i1})},$$
$$\left(q_i^{S_{i2}-S_{i1}-r} - q_i^{-(S_{i2}-S_{i1}-r)}\right)\left(q_i^{S_{i2}-S_{i1}+s-t+1} - q_i^{-(S_{i2}-S_{i1}+s-t+1)}\right)$$
$$= q_i^{2(S_{i2}-S_{i1})} - q_i^{2r} - q_i^{-2r} + q_i^{-2(S_{i2}-S_{i1})}.$$

By the definition of $I_0$, we have $S_{i2} - S_{i1} \neq 0$ or $T_{i2} - T_{i1} \neq 0$ for $i = i_2 \in I_0$. Since $r$ runs from 1 to $s$, there is an $r$ such that $r = |S_{i2} - S_{i1}|$ or $r = |T_{i2} - T_{i1}|$ for $i = i_2 \in I_0$, which implies $u \cdot v_2 = 0$.

Similarly, $u \cdot v_j = 0$ for all $j \neq 1$. It follows that $u \cdot v = v_1$, and hence $v_1 \in V^{\mathbf{A}}$. ∎

COROLLARY 3.6. *For all $\mu \in P$, $V_\mu^{\mathbf{A}}$ is a free* $\mathbf{A}$*-module, and* $\operatorname{rank}_{\mathbf{A}} V_\mu^{\mathbf{A}} = \dim_{\mathbf{F}(q)} V_\mu^q$.

*Proof.* By Proposition 3.4 and Proposition 3.5, we have an $\mathbf{F}(q)$-linear isomorphism $\mathbf{F}(q) \otimes_{\mathbf{A}} V_\mu^{\mathbf{A}} \cong V_\mu^q$ for all $\mu \in P$, and our assertion follows. ∎

## 4. CLASSICAL LIMITS

Recall that $\mathbf{A} = \mathbf{F}[q, q^{-1}, (1/\{N\}_i)\ (i \in I,\ N \in \mathbf{Z}_{>0})]$. Let $\mathbf{J}$ be the ideal of $\mathbf{A}$ generated by $q - 1$. Then we have an isomorphism of fields $\mathbf{A}/\mathbf{J} \cong \mathbf{F}$ given by $f + \mathbf{J} \mapsto f(1)$ for $f \in \mathbf{A}$. Define $U = \mathbf{F} \otimes_{\mathbf{A}} U_{\mathbf{A}}$ and $V = \mathbf{F} \otimes_{\mathbf{A}} V^{\mathbf{A}}$. Note that $U \cong U_{\mathbf{A}}/\mathbf{J}U_{\mathbf{A}}$ and $V \cong V^{\mathbf{A}}/\mathbf{J}V^{\mathbf{A}}$. For each $\mu \in P$, define $V_\mu =$

$\mathbf{F} \otimes_{\mathbf{A}} V_\mu^{\mathbf{A}}$. Since $V^{\mathbf{A}} = \bigoplus_{\mu \in P} V_\mu^{\mathbf{A}}$, we have $V = \bigoplus_{\mu \in P} V_\mu$. Moreover, we have the following.

PROPOSITION 4.1. $\dim_{\mathbf{F}} V_\mu = \operatorname{rank}_{\mathbf{A}} V_\mu^{\mathbf{A}}$.

*Proof.* If $\{v_j \mid j = 1, \ldots, m\}$ ($m = \operatorname{rank}_{\mathbf{A}} V_\mu^{\mathbf{A}}$) is a basis of the free $\mathbf{A}$-module $V_\mu^{\mathbf{A}}$, by [Hu, Theorem 5.11, Chap. 4], every element $v$ of $V_\mu = \mathbf{F} \otimes_{\mathbf{A}} V_\mu^{\mathbf{A}}$ can be written uniquely as $v = \sum_{j=1}^m a_j \otimes v_j$ ($a_j \in \mathbf{F}$). It follows that $\{\bar{v}_j = 1 \otimes v_j \mid j = 1, \ldots, m\}$ is a basis of $\mathbf{F}$-vector space $V_\mu$. ∎

Consider the natural maps $U_{\mathbf{A}} \to U_{\mathbf{A}}/\mathbf{J}U_{\mathbf{A}} \cong U$ and $V^{\mathbf{A}} \to V^{\mathbf{A}}/\mathbf{J}V^{\mathbf{A}} \cong V$. Under these natural maps, we see that $q \to 1$. The passage from $U_{\mathbf{A}}$ (resp. $V^{\mathbf{A}}$) to $U$ (resp. $V$) under these maps is referred to as taking *classical limit.* We denote by $\bar{u}$ and $\bar{v}$ the images of the elements $u \in U_{\mathbf{A}}$ and $v \in V^{\mathbf{A}}$, respectively. We also denote by $\overline{h_i}$ and $\overline{d_i}$ ($i \in I$) the images of $(K_i - K_i^{-1})/(q_i - q_i^{-1})$ and $(D_i - D_i^{-1})/(q_i - q_i^{-1})$, respectively.

LEMMA 4.2. *As endomorphisms of $V$, $\overline{q^h} = 1$ for all $h \in P^\vee$. In particular, $\overline{K_i} = \overline{D_i} = 1$ for all $i \in I$.*

*Proof.* By Proposition 4.1, if $\{v_j\}_{j=1}^m$ ($m = \operatorname{rank}_{\mathbf{A}} V_\mu^{\mathbf{A}}$) is an $\mathbf{A}$-basis of $V_\mu^{\mathbf{A}}$ ($\mu \in P$), then $\{\overline{v_j} = 1 \otimes v_j\}_{j=1}^m$ is an $\mathbf{F}$-basis of $V_\mu$. Since $q^h \cdot v_j = q^{\mu(h)} v_j$ for all $h \in P^\vee$, $j = 1, \ldots, m$, letting $q \to 1$, we obtain $\overline{q^h} \cdot \overline{v_j} = \overline{v_j}$ for all $j = 1, \ldots, m$. Therefore, $\overline{q^h} = 1$ on $V_\mu$ ($\mu \in P$) and, hence, on $V$. ∎

PROPOSITION 4.3. *As an $\mathbf{F}$-algebra of endomorphisms of $V$, $U$ is generated by $\overline{e_{ik}}$, $\overline{f_{ik}}$, $\overline{h_i}$, $\overline{d_i}$ ($i \in I$, $k = 1, \ldots, m_i$).*

*Proof.* As $q \to 1$, by (3.4), $\{N\}_i \to N$ for $i \in I$, $N \in \mathbf{Z}_{\geq 0}$. Hence $\overline{e_{ik}^{\{N\}}} = \overline{e_{ik}}^N/N!$, and $\overline{f_{ik}^{\{N\}}} = \overline{f_{ik}}^N/N!$. Recall that

$$\begin{bmatrix} K_i; c \\ N \end{bmatrix} = \prod_{r=1}^{N} \frac{K_i q_i^{c-r+1} - K_i^{-1} q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}}.$$

Note that

$$\begin{aligned}
&\frac{K_i q_i^{c-r+1} - K_i^{-1} q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}} \\
&\quad = \frac{K_i q_i^{c-r+1} - K_i^{-1} q_i^{c-r+1} + K_i^{-1} q_i^{c-r+1} - K_i^{-1} q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}} \\
&\quad = \frac{q_i^{c-r+1}(K_i - K_i^{-1})}{q_i^r - q_i^{-r}} + K_i^{-1} \frac{q_i^{c-r+1} - q_i^{-(c-r+1)}}{q_i^r - q_i^{-r}} \\
&\quad = \frac{q_i^{c-r+1}(q_i - q_i^{-1})}{q_i^r - q_i^{-r}} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} + K_i^{-1} \frac{q_i^{c-r+1} - q_i^{-(c-r+1)}}{q_i - q_i^{-1}} \frac{q_i - q_i^{-1}}{q_i^r - q_i^{-r}}.
\end{aligned}$$

Letting $q \to 1$ yields

$$\overline{\begin{bmatrix} K_i; c \\ N \end{bmatrix}} = \prod_{r=1}^{N} \frac{1}{r}(\overline{h_i} + (c - r + 1)1).$$

Similarly, we obtain

$$\overline{\begin{bmatrix} D_i; c \\ N \end{bmatrix}} = \prod_{r=1}^{N} \frac{1}{r}(\overline{d_i} + (c - r + 1)1)$$

and

$$\overline{\begin{Bmatrix} K_i; c; M \\ N \end{Bmatrix}} = \frac{\Pi_{r=1}^{N}(\overline{h_i} + (t_i(c - r + 1) + M)1)}{N!}.$$

Therefore, as an **F**-algebra of endomorphisms of $V$, $U$ is generated by $\overline{e_{ik}}$, $\overline{f_{ik}}$, $\overline{h_i}$, $\overline{d_i}$ $(i \in I,\ k = 1, \dots, m_i)$. ∎

THEOREM 4.4. (a) *The endomorphisms* $\overline{e_{ik}}$, $\overline{f_{ik}}$, $\overline{h_i}$, $\overline{d_i}$ $(i \in I,\ k = 1, \dots, m_i)$ *satisfy the relations* (1.2). *Hence $V$ has a $U(\mathfrak{g})$-module structure.*

(b) *As a $U(\mathfrak{g})$-module, $V$ is a highest weight module with highest weight $\lambda \in P$ and highest weight vector $\overline{v_\lambda} = 1 \otimes v_\lambda$.*

(c) *The endomorphisms $\overline{h_i}$ and $\overline{d_i}$ $(i \in I)$ act on $V_\mu$ $(\mu \in P)$ as scalar multiplication by $\mu(h_i)$ and $\mu(d_i)$, respectively. Hence $V_\mu$ is the $\mu$-weight space of the $U(\mathfrak{g})$-module $V$.*

*Proof.* (a) Since

$$\left[\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \frac{K_j - K_j^{-1}}{q_j - q_j^{-1}}\right] = 0 \qquad \text{for } i, j \in I,$$

by letting $q \to 1$ we obtain $[\overline{h_i}, \overline{h_j}] = 0$. Similarly, $[\overline{h_i}, \overline{d_j}] = 0$ and $[\overline{d_i}, \overline{d_j}] = 0$.

For $v \in V_\mu^{\mathbf{A}}$ $(\mu \in P)$ and $i, j \in I$, $k = 1, \dots, m_j$, we have

$$\begin{aligned}
&\left(\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} e_{jk} - e_{jk} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}\right) v \\
&= e_{jk}\left(\frac{K_i q_i^{a_{ij}} - K_i^{-1} q_i^{-a_{ij}}}{q_i - q_i^{-1}} - \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}\right) v \\
&= e_{jk}\left(\frac{q_i^{\mu(h_i) + a_{ij}} - q_i^{-(\mu(h_i) + a_{ij})}}{q_i - q_i^{-1}} - \frac{q_i^{\mu(h_i)} - q_i^{-\mu(h_i)}}{q_i - q_i^{-1}}\right) v.
\end{aligned}$$

Thus, by letting $q \to 1$, it follows that

$$\begin{aligned}(\overline{h_i}\,\overline{e_{jk}} - \overline{e_{jk}}\,\overline{h_i}) \cdot \overline{v} &= (\mu(h_i) + a_{ij} - \mu(h_i))\overline{e_{jk}} \cdot \overline{v} \\ &= a_{ij}\overline{e_{jk}}\overline{v}.\end{aligned}$$

Therefore, $\overline{h_i}\,\overline{e_{jk}} - \overline{e_{jk}}\,\overline{h_i} = a_{ij}\overline{e_{jk}}$ on $V$ for $i, j \in I$, $k = 1, \dots, m_j$. Similarly, we obtain

$$\begin{aligned}\overline{h_i}\,\overline{f_{jk}} - \overline{f_{jk}}\,\overline{h_i} &= -a_{ij}\overline{f_{jk}}, \\ \overline{d_i}\,\overline{e_{jk}} - \overline{e_{jk}}\,\overline{d_i} &= \delta_{ij}\overline{e_{jk}}, \\ \overline{d_i}\,\overline{f_{jk}} - \overline{f_{jk}}\,\overline{d_i} &= -\delta_{ij}\overline{f_{jk}}.\end{aligned}$$

The rest of the relations are immediate consequences of the definition of the endomorphisms $\overline{e_{jk}}, \overline{f_{jk}}, \overline{h_i}, \overline{d_i}$ and the fact that $[m]_i! \to m!$ as $q \to 1$ $(m \in \mathbf{Z}_{\geq 0})$.

(b) It is clear that $\overline{e_{ik}} \cdot \overline{v_\lambda} = 0$ for $i \in I$, $k = 1, \dots, m_i$. Since

$$\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} v_\lambda = \frac{q_i^{\lambda(h_i)} - q_i^{-\lambda(h_i)}}{q_i - q_i^{-1}} v_\lambda,$$

letting $q \to 1$, we obtain $\overline{h_i} \cdot \overline{v_\lambda} = \lambda(h_i)\overline{v_\lambda}$ for $i \in I$. Similarly, we have $\overline{d_i} \cdot \overline{v_\lambda} = \lambda(d_i)\overline{v_\lambda}$ for $i \in I$. Recall that $V^{\mathbf{A}} = U_{\mathbf{A}}^- \cdot v_\lambda$, where $U_{\mathbf{A}}^-$ is the $\mathbf{A}$-subalgebra of $U_{\mathbf{A}}$ generated by the elements $f_{ik}^{(N)}$ for $i \in I$, $k = 1, \dots, m_i$. Hence $V = U^- \cdot \overline{v_\lambda}$, where $U^-$ is the $\mathbf{F}$-subalgebra of $U$ with 1 generated by the elements $\overline{f_{ik}}^N/N!$, equivalently, generated by $\overline{f_{ik}}$ for $i \in I$, $k = 1, \dots, m_i$. Therefore, $V$ is a highest weight module over $U(\mathfrak{g})$ with highest weight $\lambda$ and highest weight vector $\overline{v_\lambda}$.

(c) For $v \in V_\mu^{\mathbf{A}}$ $(\mu \in P)$ and $i \in I$, we have

$$\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} v = \frac{q_i^{\mu(h_i)} - q_i^{-\mu(h_i)}}{q_i - q_i^{-1}} v,$$

which yields $\overline{h_i} \cdot \overline{v} = \mu(h_i)\overline{v}$ as $q \to 1$. Similarly, $\overline{d_i} \cdot \overline{v} = \mu(d_i)\overline{v}$ $(i \in I)$. ∎

We now prove our main results.

THEOREM 4.5. *If $V^q$ is the Verma module $M^q(\lambda)$ over $U_q(\mathfrak{g})$ with highest weight $\lambda \in P$, then $V$ is isomorphic to the Verma module $M(\lambda)$ over $U(\mathfrak{g})$ with highest weight $\lambda$. Therefore, any Verma module $M(\lambda)$ over $U(\mathfrak{g})$ with highest weight $\lambda \in P$ admits a quantum deformation to the Verma module $M^q(\lambda)$ over $U_q(\mathfrak{g})$ with highest weight $\lambda \in P$ in such a way that the dimensions of weight spaces are invariant under the deformation.*

*Proof.* Let $v_\lambda$ be a highest weight vector of $V^q$. It suffices to prove that $V$ is a free $U^-$-module of rank one generated by the vector $\overline{v_\lambda}$. Since $V^q = M^q(\lambda)$, $V^q$ is a free $U_q(\mathfrak{g}^-)$-module of rank one generated by the highest weight vector $v_\lambda$. Thus by Proposition 3.3 and Proposition 3.4, the $\mathbf{A}$-form $V^{\mathbf{A}}$ of $V^q$ is also a free $U_{\mathbf{A}}^-$-module generated by $v_\lambda$. Since $V^{\mathbf{A}} = U_{\mathbf{A}}^- \cdot v_\lambda$, by letting $q \to 1$, we have $V = U^- \cdot \overline{v_\lambda}$, where $U^- = \mathbf{F} \otimes_{\mathbf{A}} U_{\mathbf{A}}^- \cong U_{\mathbf{A}}^- / \mathbf{J} U_{\mathbf{A}}^-$ is the $\mathbf{F}$-subalgebra of $U$ with 1 generated by $f_{ik}$ for $i \in I$, $k = 1, \ldots, m_i$. Suppose $\bar{u} \cdot \overline{v_\lambda} = 0$ for some $\bar{u} \in U^-$. Write $\bar{u} = u + \mathbf{J}U_{\mathbf{A}}^-$ for some $u \in U_{\mathbf{A}}^-$. Since $\bar{u} \cdot \overline{v_\lambda} = 0$, $u \cdot v_\lambda \in \mathbf{J}V^{\mathbf{A}} = \mathbf{J}U_{\mathbf{A}}^- \cdot v_\lambda$. So $u \cdot v_\lambda = u' \cdot v_\lambda$ for some $u' \in \mathbf{J}U_{\mathbf{A}}^-$. Since $V^{\mathbf{A}}$ is a free $U_{\mathbf{A}}^-$-module generated by $v_\lambda$, we must have $u = u'$, which implies $\bar{u} = 0$ in $U^-$. Therefore, $V$ is a free $U^-$-module generated by $\overline{v_\lambda}$ and, hence, $V \cong M(\lambda)$. The second assertion follows from Corollary 3.6 and Proposition 4.1. ∎

THEOREM 4.6. *If $\lambda \in P^+$ and $V^q$ is the irreducible highest weight module $V^q(\lambda)$ over $U_q(\mathfrak{g})$ with highest weight $\lambda$, then $V$ is isomorphic to the irreducible highest weight module $V(\lambda)$ over $U(\mathfrak{g})$ with highest weight $\lambda$. Therefore, any irreducible highest weight module $V(\lambda)$ over $U(\mathfrak{g})$ with highest weight $\lambda \in P^+$ admits a quantum deformation to the irreducible highest weight module $V^q(\lambda)$ over $U_q(\mathfrak{g})$ with highest weight $\lambda \in P^+$ in such a way that the dimensions of weight spaces are invariant under the deformation. In particular, the character of $V^q(\lambda)$ is given by the Weyl–Kac–Borcherds formula.*

*Proof.* Let $v_\lambda$ be a highest weight vector of $V^q = V^q(\lambda)$. By Corollary 3.2, if $\lambda(h_i) = 0$, then $f_{ik} \cdot v_\lambda = 0$ for $k = 1, \ldots, m_i$, and if $a_{ii} = 2$, then $f_{ik}^{\lambda(h_i)+1} \cdot v_\lambda = 0$. Letting $q \to 1$, we see that $V$ is a highest weight module over $U(\mathfrak{g})$ with highest weight $\lambda \in P^+$ and highest weight vector $\overline{v_\lambda}$ that satisfies the conditions of Proposition 1.3. Therefore, $V$ is isomorphic to $V(\lambda)$. The second assertion follows from Corollary 3.6 and Proposition 4.1. ∎

COROLLARY 4.7. *If $\lambda \in P^+$ and $V^q$ is a highest weight module over $U_q(\mathfrak{g})$ with highest weight $\lambda$ and highest weight vector $v_\lambda$ such that*

(i) *if $\lambda(h_i) = 0$, then $f_{ik} \cdot v_\lambda = 0$ for $k = 1, \ldots, m_i$,*
(ii) *if $a_{ii} = 2$, then $f_{ik}^{\lambda(h_i)+1} \cdot v_\lambda = 0$,*

*then $V^q$ is isomorphic to the irreducible highest weight module $V^q(\lambda)$.*

*Proof.* As we have seen in the proof of Theorem 4.6, if $V^q$ is a highest weight $U_q(\mathfrak{g})$-module satisfying the above conditions, then $V = \mathbf{F} \otimes_{\mathbf{A}} V^{\mathbf{A}}$ is a highest weight $U(\mathfrak{g})$-module satisfying the conditions of Proposition

1.3. Hence $V \cong V(\lambda)$, and $\operatorname{ch} V$ is given by the Weyl–Kac–Borcherds formula. By Corollary 3.6 and Proposition 4.1, we have

$$\operatorname{Ch} V^q = \operatorname{ch} V = \operatorname{ch} V(\lambda) = \operatorname{Ch} V^q(\lambda).$$

Therefore we have $V^q \cong V^q(\lambda)$. ∎

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